Published by Institute of Physics Publishing for SISSA/ISAS
Received: *November 6, 2002*
Revised: *August 29, 2003*
Accepted: *November 11, 2003*

The boundary state formalism and conformal invariance in off-shell string theory

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Abstract: In this note we present a generalization of the boundary state formalism for the bosonic string that allows us to calculate the overlap of the boundary state with arbitrary closed string states. We show that this generalization exactly reproduces world-sheet sigma model calculations, thus giving the correct overlap with both on- and off-shell string states, and that this new boundary state automatically satisfies the requirement for integrated vertex operators in the case of non-conformally invariant boundary interactions

Keywords: Bosonic Strings, D-branes, Tachyon Condensation.

JHEP11(2003)021

Contents

1. Introduction

The problem of studying off-shell string theory is an old one, and there have been many attempts to examine it. A particularly interesting approach is background independent string field theory [1]–[5] which has received attention recently as an approach which facilitates the understanding of properties of unstable d-branes [6]–[8]. A tractable problem that can be approached within this formalism is to understand the behavior of the off-shell theory in the background of a tachyon field, in particular a quadratic function of the coordinates. This model has been the subject of some research interest [6]–[18] and the resulting theory is not invariant under conformal transformations except in the trivial cases of vanishing or infinite quadratic tachyon potential. This subtlety reveals an interesting structure. In this note we suggest a generalization of the boundary state [19]–[24] which naturally accommodates this loss of invariance. This new boundary state can be applied to the problem of computing off-shell closed string emission amplitudes from a d-brane.

For definiteness, we consider the following action on the string world sheet

$$\begin{aligned} S(g,F,T_0,U) = &\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma g_{\mu\nu}\partial^a X^\mu \partial_a X^\nu + \\ &+\int_{\partial\Sigma} d\theta \left(\frac{1}{2}F_{\mu\nu}X^\nu\partial_t X^\mu + \frac{1}{2\pi}T_0 + \frac{1}{8\pi}U_{\mu\nu}X^\mu X^\nu\right). \end{aligned} \tag{1.1}$$

Here, α' is the inverse string tension, Σ is the string world sheet with measure $d^2\sigma$, $\partial\Sigma$ is world sheet boundary with measure $d\theta$, and ∂_t is the tangential derivative to the boundary. The background fields that are included in this are $F_{\mu\nu}$, a constant gauge field strength, and the tachyon profile parameterized by a constant and a symmetric matrix, $T(X) = \frac{1}{2\pi}T_0 + \frac{1}{8\pi}U_{\mu\nu}X^\mu X^\nu$. We choose this form of boundary interaction because the quadratic nature makes the world sheet sigma model exactly solvable, and thus facilitates comparison between boundary state calculations and sigma model results. The quadratic tachyon presented above is a simple re-writing of the action studied in [2], and has the additional interesting feature that it interpolates between Neumann and Dirichlet boundary conditions for the world sheet fields X^μ. The constant gauge field strength is added to the world sheet action because the sigma model consequences of this are known both at the open string tree and one loop level [25], and it is also interesting from the more recent study of the decay of d-branes with electric flux [26].

The organization of this note is as follows. First we calculate the expectation values for single vertex operators using the string world sheet action (1.1), and, because the presence of the tachyon field breaks the conformal invariance of the string world sheet, we calculate the effect of varying the insertion point of the vertex operator. We then calculate the boundary state for the action (1.1), and show that by averaging this boundary state over PSL$(2, R)$ transformations of the string world sheet we reproduce the sigma model results, indicating that this is an appropriate boundary state for the theory: When the boundary conditions act as operator equations on the boundary state they annihilate it [27], and the boundary state correctly reproduces the one point functions for closed strings [28]. Finally, we apply this technique to the problem of the one loop partition functions in the spirit of [27]. Our calculation gives an explicit realization of the proposal [12] for open string one loop partition functions with non-conformally invariant boundary interactions, we verify that in the case of a constant gauge field strength we obtain the known result [25], and we show that these one loop effects do not change the stability conditions for Neumann and Dirichlet boundary conditions.

2. Sigma model calculations

We commence by calculating some closed string emission amplitudes on the disk world sheet, because the boundary state will be seen to emit closed string modes in the same way. It is useful to fix conventions: the functional integral which we compute is in all cases an average over the action (1.1),

$$\langle \mathcal{O}(X) \rangle = \int \mathcal{D}X e^{-S(X)} \mathcal{O}(X) \,. \tag{2.1}$$

Throughout, we will use $z = \rho e^{i\phi}$ as a parameterization of the points within the unit disk, so $0 \leq \rho \leq 1$ and $0 \leq \phi < 2\pi$. In addition, the Greens function on the unit disk with Neumann boundary conditions is determined to be [29]

$$G^{\mu\nu}(z, z') = -\alpha' g^{\mu\nu} \left(-\ln |z - z'| + \ln |1 - z\bar{z}'| \right) , \tag{2.2}$$

from which the bulk to boundary propagator can be determined

$$G^{\mu\nu}(\rho e^{i\phi}, e^{i\phi'}) = 2\alpha' g^{\mu\nu} \sum_{m=1}^{\infty} \frac{\rho^m}{m} \cos[m(\phi - \phi')] \,. \tag{2.3}$$

The boundary to boundary propagator can be read off from (2.3) as the limit in which $\rho \to 1$. Using the bulk to boundary propagator it is possible to integrate out the quadratic interactions on the boundary [25] to obtain a modified propagator

$$\begin{aligned}
G^{\mu\nu}(z,z') &= \alpha' g^{\mu\nu}\ln\left|z-z'\right| - \alpha' g^{\mu\nu}\ln\left|1-z\bar{z}'\right| - \\
&\quad -\alpha'\sum_{n=1}^{\infty}\left(\frac{2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{n}}\right)^{\mu\nu}\frac{(z\bar{z}')^n+(\bar{z}z')^n}{n} \\
&= \alpha' g^{\mu\nu}\ln\left|z-z'\right| + \frac{\alpha'}{2}\sum_{n=1}^{\infty}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{n}}\right)^{\mu\nu}\frac{(z\bar{z}')^n+(\bar{z}z')^n}{n}\,.
\end{aligned}$$

We now wish to calculate the disk partition function. In the sigma model approach the oscillator modes of X must be integrated out with the contributions from F and U treated as perturbations. Since both perturbations are quadratic, all the Feynmann graphs that contribute to the free energy can be written and evaluated, and because the interaction terms F and U are restricted to the boundary we find

$$\begin{aligned}
\mathcal{F} &= \sum_{n=1}^{\infty}\frac{1}{n}\int d\phi_1\cdots d\phi_n(-1)^n\left[\left(F_{\mu_1\nu_1}\partial_{\phi_1}+\frac{1}{4\pi}U_{\mu_1\nu_1}\right)2g^{\nu_1\mu_2}\sum_{m_1=1}^{\infty}\frac{\cos[m_1(\phi_1-\phi_2)]}{m_1}\times\cdots\right. \\
&\qquad\qquad \left.\cdots\times\left(F_{\mu_n\nu_n}\partial_{\phi_n}+\frac{1}{4\pi}U_{\mu_n\nu_n}\right)2g^{\nu_n\mu_1}\sum_{m_n=1}^{\infty}\frac{\cos[m_n(\phi_n-\phi_1)]}{m_n}\right] \\
&= -\sum_{m=1}^{\infty}\mathrm{Tr}\ln\left(g+2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{m}\right). \qquad (2.4)
\end{aligned}$$

(See [25, 30] for examples of calculations done in this spirit.) From (2.4) we obtain the partition function

$$\begin{aligned}
Z &= e^{-T_0}\prod_{m=1}^{\infty}\frac{1}{\det\left(g+2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{m}\right)}\int dx_0 e^{-\frac{U_{\mu\nu}}{4}x_0^\mu x_0^\nu} \\
&= \frac{1}{\det\left(U/2\right)}e^{-T_0}\prod_{m=1}^{\infty}\frac{1}{\det\left(g+2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{m}\right)}. \qquad (2.5)
\end{aligned}$$

This expression is divergent, but using ζ-function regularization [11] it can be reduced to

$$Z = e^{-T_0}\sqrt{\det\left(\frac{g+2\pi\alpha' F}{U/2}\right)}\det\Gamma\left(1+\frac{\alpha' U/2}{g+2\pi\alpha' F}\right), \qquad (2.6)$$

where $\Gamma(g)$ is the Γ function and the dependence of all transcendental functions on the matrices U and F is defined by their Taylor expansion.

We now wish to calculate the expectation value for vertex operators that correspond to different closed string states, however this is a process that must be done with some care. To calculate the emission of a closed string in the world-sheet picture one generally considers a disk emitting an asymptotic closed string state. This is really a closed string cylinder diagram. The standard method is to use conformal invariance to map the closed string state to a point on the disk, namely the origin, where a corresponding vertex operator is inserted. On the other hand it has been argued that it is necessary to have an integrated vertex operator for closed string states to properly couple [12], in particular that the graviton must be produced by an integrated vertex operator to couple correctly

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to the energy momentum tensor. The distinction between a fixed vertex operator and an integrated vertex operator is moot in the conformally invariant case where the integration will only produce a trivial volume factor, however in the case we consider more care must be taken. We wish to consider arbitrary locations of the vertex operators on the string world sheet, and the natural measure to impose is that of the conformal transformations which map the origin to a point within the unit disk on the complex plane.

In other words we propose to allow the vertex operator corresponding to the closed string state to be moved from the origin by a conformal transformation that preserves the area of the unit disk, namely a PSL$(2, R)$ transformation. The method to accomplish this is to go to a new coordinate system

$$y = \frac{az+b}{\bar{b}z+\bar{a}}, \qquad |a^2| - |b^2| = 1\,, \tag{2.7}$$

and a vertex operator at the origin $y = 0$ would correspond to an insertion of a vertex operator at the point $z = -b/a$. We note that in the only change to the Greens function is the addition of a term that is harmonic within the unit disk. The parameter of the integration over the position of the vertex operator would be to the measure on PSL$(2, R)$, giving an infinite factor in the conformally invariant case [4, 12, 31]. From this, we have a definite prescription for the calculation of vertex operator expectation values: use the conformal transformation to modify the Greens function, and calculate the expectation values of operators at the origin with this modified Greens function. It was shown explicitly in [24] that for the case of a conformally invariant boundary interaction the Haar measure on PSL$(2, R)$ is required to render the expectation value of closed string vertex operators invariant under such a transformation.

Now we use this prescription to calculate the sigma model expectation values of some operators. We will start with the closed string tachyon vertex operator, $:e^{ip_\mu X^\mu(z(y))}:$ inserted at the point $y = 0$. The normal ordering prescription for all such operators is that any divergent pieces will be subtracted, but finite pieces will remain and by inspection the appropriately subtracted Greens function is

$$\begin{aligned} :\mathcal{G}^{\mu\nu}(z,z'): &= G^{\mu\nu}(z,z') - g^{\mu\nu}\alpha' \ln|z-z'| \\ &= \frac{\alpha'}{2}\sum_{n=1}^{\infty}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{n}}\right)^{\mu\nu}\frac{(z\bar{z}')^n+(\bar{z}z')^n}{n}\,. \end{aligned} \tag{2.8}$$

Using (2.8) we obtain

$$\begin{aligned} \langle :e^{ip_\mu X^\mu(y=0)}:\rangle &= Ze^{-\frac{1}{2}p_\mu p_\nu :\mathcal{G}^{\mu\nu}(z(y),z'(y)):}\Big|_{y=0} \\ &= Z\exp\left(-\frac{\alpha'}{2}p_\mu p_\nu \sum_{n=1}^{\infty}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{n}}\right)^{\mu\nu}\frac{1}{n}\frac{|b^{2n}|}{|a^{2n}|}\right), \end{aligned}$$

and so we find that the expectation value for the tachyon vertex operator is

$$\begin{aligned} &\int d^2a d^2b\,\delta(|a^2|-|b^2|-1)\langle :e^{ip_\mu X^\mu(y(a,b)=0)}:\rangle = \\ &= \int d^2a d^2b\,\delta(|a^2|-|b^2|-1) Z\exp\left(-\frac{\alpha'}{2}p_\mu p_\nu \sum_{n=1}^{\infty}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{n}}\right)^{\mu\nu}\frac{1}{n}\frac{|b^{2n}|}{|a^{2n}|}\right). \end{aligned} \tag{2.9}$$

We perform a similar calculation for the massless closed string excitations. In particular the graviton insertion at $y = 0$ is given by

$$\langle \mathcal{V}_h \rangle = \left\langle : -\frac{2}{\alpha'} h_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu e^{ip_\mu X^\mu(y=0)} : \right\rangle \tag{2.10}$$

where h is a symmetric traceless tensor and the normalization follows the conventions of [32]. We use the same techniques as for the tachyon, and note that there will be cross contractions between the exponential and the X-field prefactors, and obtain

$$\begin{aligned}
\langle \mathcal{V}_h \rangle &= -\frac{2}{\alpha'} Z h_{\mu\nu} \Big(\partial \bar{\partial}' : \mathcal{G}^{\mu\nu} \left(z(y), z'(y)\right) : + \partial : \mathcal{G}^{\mu\alpha} \left(z(y), z'(y)\right) : \times \\
&\qquad \times \bar{\partial} : \mathcal{G}^{\mu\beta} \left(z(y), z'(y)\right) : (ip_\alpha)(ip_\beta) \Big) e^{-\frac{1}{2} p_\mu p_\nu : \mathcal{G}^{\mu\nu}(z(y), z'(y)):}\Big|_{y=0} \\
&= Z h_{\mu\nu} \Bigg(-\sum_{n=1}^{\infty} \left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}} \right)^{\mu\nu} n \frac{|b^{2(n-1)}|}{|a^{2(n-1)}|} \frac{1}{|a^2|^2} + \\
&\qquad + \frac{\alpha'}{2} \sum_{n=1}^{\infty} \left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}} \right)^{\mu\alpha} \frac{|b^{2(n-1)}|}{|a^{2(n-1)}|} \frac{-b}{|a^2| a} \times \\
&\qquad \times \sum_{m=1}^{\infty} \left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{m}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{m}} \right)^{\nu\beta} \frac{|b^{2(m-1)}|}{|a^{2(m-1)}|} \frac{-\bar{b}}{|a^2| \bar{a}} p_\alpha p_\beta \Bigg) \times \\
&\qquad \times \exp\left(-\frac{\alpha'}{2} p_\mu p_\nu \sum_{n=1}^{\infty} \left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}} \right)^{\mu\nu} \frac{1}{n} \frac{|b^{2n}|}{|a^{2n}|} \right).
\end{aligned} \tag{2.11}$$

The results for either the Kalb-Ramond field or the dilaton are obtained by replacing $h_{\mu\nu}$ with the appropriate polarization tensor for either field in the above. As in (2.9) the expectation value for the graviton operator is given by integrating (2.11) over $\mathrm{PSL}(2, R)$.

Finally, we perform the same kind of calculation for a more general closed string state. We consider a state which may be off shell in the sense that it not annihilated by the positive modes of the σ-model energy momentum tensor (the Virasoro generators), may not satisfy the mass shell condition, and may not be level matched. Our explicit choice is to consider the operator

$$\langle \mathcal{V}_A \rangle = \left\langle : -i \left(\frac{2}{\alpha'} \right)^{3/2} A_{\mu\nu\delta} \frac{\partial^a}{(a-1)!} X^\mu \frac{\bar{\partial}^b}{(b-1)!} X^\nu \frac{\bar{\partial}^c}{(c-1)!} X^\gamma e^{ip_\mu X^\mu} : \right\rangle \tag{2.12}$$

which is an arbitrary state involving three creation operators. We find that

$$\begin{aligned}
\langle \mathcal{V}_A \rangle &= Z A_{\mu\nu\delta} \left(\frac{2}{\alpha'} \right)^{3/2} \Bigg(\frac{\partial^a}{(a-1)!} \frac{\bar{\partial}'^b}{(b-1)!} : G^{\mu\nu}(z, z') : \frac{\bar{\partial}^c}{(c-1)!} : G^{\delta\alpha}(z, z') : p_\alpha + \\
&\qquad + \frac{\partial^a}{(a-1)!} \frac{\bar{\partial}'^c}{(c-1)!} : G^{\mu\delta}(z, z') : \frac{\bar{\partial}^b}{(b-1)!} : G^{\nu\alpha}(z, z') : p_\alpha - \\
&\qquad - \frac{\partial^a}{(a-1)!} : G^{\mu\alpha}(z, z') : \frac{\bar{\partial}^b}{(b-1)!} : G^{\nu\beta}(z, z') : \frac{\bar{\partial}^c}{(c-1)!} : \\
&\qquad : G^{\delta\gamma}(z, z') : p_\alpha p_\beta p_\gamma \Bigg) e^{-\frac{1}{2} p_\mu p_\nu : \mathcal{G}^{\mu\nu}(z, z'):}\Bigg|_{y=0}.
\end{aligned} \tag{2.13}$$

It is straightforward but not very instructive to simplify this to obtain results like (2.9) and (2.11), and in the next section we obtain a result that simplifies these calculations. This state will allow us to check the consistency of the prescription for the boundary state presented in the next section.

3. Boundary states

We now perform the same kind of analysis from the point of view of the boundary state formalism. To this end, it is important to understand where the boundary state comes from. By varying the action (1.1) one obtains the equation

$$\frac{1}{2\pi\alpha'} g_{\mu\nu}\partial_n X^\nu + F_{\mu\nu}\partial_t X^\nu + \frac{1}{4\pi} U_{\mu\nu} X^\nu = 0 \tag{3.1}$$

on the boundary of the string world-sheet. As discussed in, for example, [20, 23] the boundary state $|B\rangle$ is a state of closed string theory which obeys the boundary condition (3.1) as an operator equation. Using the standard mode expansion for X as a function of the world sheet coordinate z

$$X^\mu(z,\bar{z}) = x^\mu + p^\mu \ln|z^2| + \sum_{m\neq 0} \frac{1}{m}\left(\frac{\alpha^\mu_m}{z^m} + \frac{\tilde{\alpha}^\mu_m}{\bar{z}^m}\right), \tag{3.2}$$

the boundary condition (3.1) can be expressed as

$$\left(g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}\right)_{\mu\nu} \alpha^\mu_n + \left(g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}\right)_{\mu\nu} \tilde{\alpha}^\mu_{-n} = 0\,. \tag{3.3}$$

The boundary state obeys (3.3) as an operator equation,

$$\begin{aligned} \left[\left(g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}\right)_{\mu\nu} \alpha^\mu_n + \left(g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}\right)_{\mu\nu} \tilde{\alpha}^\mu_{-n}\right] |B\rangle &= 0\,, \\ \left[g_{\mu\nu} p^\mu - i\frac{\alpha'}{2} U_{\mu\nu} x^\mu\right] |B\rangle &= 0\,. \end{aligned} \tag{3.4}$$

The state that satisfies this must be a coherent state and is given by

$$|B\rangle = \mathcal{N} \prod_{n\geq 1} \exp\left(-\left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}}\right)_{\mu\nu} \frac{\alpha^\mu_{-n}\tilde{\alpha}^\nu_{-n}}{n}\right) \exp\left(-\frac{\alpha'}{4} x^\mu U_{\mu\nu} x^\nu\right)|0\rangle\,, \tag{3.5}$$

where $\mathcal{N}$ is an as yet undetermined normalization constant.

We now examine how this boundary state transforms under the action of the residual conformal symmetry of the disk, namely under $\mathrm{PSL}(2,R)$ transformations. In the two conformally invariant cases ($U = 0$ and $U = \infty$) this is a good symmetry of the action, but we expect there to be some interpolation as the flow from $U = 0$ to $U = \infty$ takes us from Neumann to Dirichlet boundary conditions. As mentioned previously the action of $\mathrm{PSL}(2,R)$ on the complex coordinates of the disk is to perform the mapping

$$z \to w(z) = \frac{az + b}{\bar{b}z + \bar{a}}\,, \tag{3.6}$$

where a and b satisfy the relation

$$|a^2| - |b^2| = 1\,. \tag{3.7}$$

This transformation maps the interior of the unit disk to itself, the exterior to the exterior and the boundary to the boundary. Moreover, this transformation of the coordinates induces a mapping which intermixes the oscillator modes. To see this consider the definition of the oscillator modes

$$\alpha_m^\mu = \sqrt{\frac{2}{\alpha'}}\oint \frac{dz}{2\pi} z^m \partial X^\mu(z)\,, \tag{3.8}$$

where the contour is the boundary of the unit disk, and the mode expansion of X gives

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_m \frac{\alpha_m^\mu}{z^{m+1}}\,. \tag{3.9}$$

Now, using the fact that X is a scalar, or equivalently the fact that ∂X is a (1,0) tensor, we see that

$$\alpha_m^\mu = \oint \frac{dz}{2\pi i} z^m \partial_w X^\mu(w)\frac{dw}{dz}\,. \tag{3.10}$$

Now, using the fact that a mode expansion for X exists in terms of w with coefficients α'_m in exactly the same way as (3.9), we see that

$$\alpha_m^\mu = M_{mn}\alpha_n^{\prime\mu}\,, \tag{3.11}$$

where

$$M_{mn} = \oint \frac{dz}{2\pi i} z^m \frac{(\bar{b}z+\bar{a})^{n-1}}{(az+b)^{n+1}}\,. \tag{3.12}$$

Upon evaluation it becomes clear that this matrix has a block diagonal form, which is due to the fact that there are no poles inside (outside) the integration region when $n < 0 < m$ ($m < 0 < n$). The consequence of this is that under PSL$(2,R)$ transformations the creation operators transform into creation operators and the annihilation operators likewise transform into annihilation operators. Furthermore, upon rescaling $\mathcal{M}_{mp} \to \sqrt{\frac{p}{m}}M_{mp}$ so that the oscillators are normalized as creation and annihilation operators it is easy to show that for both positive and negative p and m

$$\mathcal{M}^{-1}_{mp} = \oint \sqrt{\frac{m}{p}}\frac{d\bar{z}}{-2\pi i}\bar{z}^p\frac{(a+b\bar{z})^{m-1}}{(\bar{b}+\bar{a}\bar{z})^{m+1}} = \bar{\mathcal{M}}^T_{mp} = \mathcal{M}^\dagger_{mp}\,. \tag{3.13}$$

This is simply a statement of the fact that M preserves the inner product on the space of operators. It would be inappropriate for M itself to be hermitean because the commutation relation between the modes is

$$[\alpha_n^\mu, \alpha_m^\nu] = n\delta_{n-m}g^{\mu\nu}\,. \tag{3.14}$$

The general expression for M when both indices are positive can easily be found by explicit contour integration and is

$$M_{mp} = \sqrt{\frac{p}{m}}\sum_{k=0}^{p}\binom{m}{k}\binom{p-1}{p-k}(-1)^{m-k}b^m|b^2|^{-k}a^{-p}\,, \tag{3.15}$$

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where the binomial coefficients $\binom{a}{b}$ are understood to vanish in all cases where $b > a$. It will be important to note that there is a non-zero overlap with the the zero mode which will be important when we make a correspondence between the sigma model calculation and the boundary state.

To accommodate the coordinate transformation under PSL$(2,R)$ the boundary state becomes

$$
\begin{aligned}
|B_{a,b}\rangle &= \mathcal{N} \exp\left(\sum_{n=1,j,k=-\infty}^{\infty} \alpha^{\mu}_{-k} M_{-n-k} \left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha}{2}\frac{U}{n}} \right)_{\mu\nu} \frac{1}{n} \bar{M}_{-n-j} \tilde{\alpha}'^{\nu}_{-j} \right) \times \\
&\quad \times \exp\left(-\frac{\alpha'}{4} x^{\mu} U_{\mu\nu} x^{\nu} \right) |0\rangle . \qquad (3.16)
\end{aligned}
$$

In this equation and all following ones we drop the ' associated with the PSL$(2,R)$ transformation for notational simplicity. We conjecture that the proper definition of the boundary state to give the correct overlap with all closed string states is

$$
|B\rangle = \int d^2a d^2b \delta(|a^2| - |b^2| - 1) |B_{a,b}\rangle . \qquad (3.17)
$$

This is just the boundary state (3.16) integrated over the Haar measure of PSL$(2,R)$. The boundary state is thus a linear combination of an infinite tower of closed string states with definite coefficients. We now verify that it gives the correct overlap with the tachyon and massless states by comparing with the sigma model calculations of the previous section.

The computation for the tachyon fixes the normalization. We take the inner product

$$
\begin{aligned}
\langle 0|B\rangle &= \mathcal{N} \int d^2a d^2b \delta(|a^2| - |b^2| - 1) \exp\left(-\frac{\alpha'}{2} x^{\mu} U_{\mu\nu} x^{\nu} \right) \times \\
&\quad \times \exp\left(-\sum_{n=1}^{\infty} \frac{\alpha'}{2} p_{\mu} M_{-n0} \frac{1}{n} \left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha}{2}\frac{U}{n}} \right)_{\mu\nu} \bar{M}_{-n0} p_{\nu} \right) \qquad (3.18)
\end{aligned}
$$

and we have used the relations $\alpha_0^{\mu} = \sqrt{\frac{\alpha'}{2}} p^{\mu}$ and $[x^{\mu}, p^{\nu}] = i g^{\mu\nu}$. We insert the explicit form $M_{-n0} = (-\bar{b}/\bar{a})^n$ to (3.18) and find

$$
\begin{aligned}
\langle 0|B\rangle &= \mathcal{N} \int d^2a d^2b \delta(|a^2| - |b^2| - 1) \exp\left(-\frac{\alpha'}{2} x^{\mu} U_{\mu\nu} x^{\nu} \right) \times \\
&\quad \times \exp\left(-\sum_{n=1}^{\infty} \frac{\alpha'}{2} p_{\mu} \frac{1}{n} \left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha}{2}\frac{U}{n}} \right)_{\mu\nu} p_{\nu} \frac{|b^{2n}|}{|a^{2n}|} \right) . \qquad (3.19)
\end{aligned}
$$

By comparing this to (2.9) we can unambiguously fix the normalization as

$$
\mathcal{N} = Z . \qquad (3.20)
$$

We now perform an analogous calculation for the massless states which will provide a non-trivial check of this normalization scheme. For an arbitrary massless state with polarization tensor $P_{\mu\nu}$

$$
|P_{\mu\nu}\rangle = P_{\mu\nu} \alpha^{\mu}_{-1} \tilde{\alpha}^{\nu}_{-1} |0\rangle \qquad (3.21)
$$

the overlap to be calculated is

$$
\begin{aligned}
\langle P_{\mu\nu}|B\rangle &= \mathcal{N}\int d^2a d^2b\delta(|a^2|-|b^2|-1)\exp\left(-\frac{\alpha'}{2}x^\mu U_{\mu\nu}x^\nu\right)\times \\
&\quad\times\exp\left(-\sum_{n=1}^{\infty}\frac{\alpha'}{2}p_\mu M_{-n0}\frac{1}{n}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{n}}\right)_{\mu\nu}\bar{M}_{-n0}p_\nu\right)\times \\
&\quad\times P_{\mu\nu}\left[-\sum_{n=1}^{\infty}M_{-n-1}\frac{1}{n}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{n}}\right)^{\mu\nu}\bar{M}_{-n-1}+\right. \\
&\qquad+\frac{\alpha'}{2}\sum_{n=1}^{\infty}M_{-n-1}\frac{1}{n}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{n}}\right)^{\mu\alpha}\bar{M}_{-n0}p_\alpha\times \\
&\qquad\left.\times\sum_{m=1}^{\infty}M_{-m0}\frac{1}{m}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{m}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{m}}\right)^{\beta\nu}\bar{M}_{-n-1}p_\beta\right] \\
&= \mathcal{N}\int d^2a d^2b\delta(|a^2|-|b^2|-1)\exp\left(-\frac{\alpha'}{2}x^\mu U_{\mu\nu}x^\nu\right)\times \\
&\quad\times\exp\left(-\sum_{n=1}^{\infty}\frac{\alpha'}{2}p_\mu\frac{1}{n}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{n}}\right)_{\mu\nu}p_\nu\frac{|b^{2n}|}{|a^{2n}|}\right)\times \\
&\quad\times P_{\mu\nu}\left[-\sum_{n=1}^{\infty}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{n}}\right)^{\mu\nu}n\frac{|b^{2(n-1)}|}{|a^{2(n-1)}|}\frac{1}{|a^2|^2}+\right. \\
&\qquad+\frac{\alpha'}{2}\sum_{n=1}^{\infty}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{n}}\right)^{\mu\alpha}\frac{|b^{2(n-1)}|}{|a^{2(n-1)}|}\frac{-\bar{b}}{|a^2|\bar{a}}p_\alpha\times \\
&\qquad\left.\times\sum_{m=1}^{\infty}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{m}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{m}}\right)^{\beta\nu}\frac{|b^{2(m-1)}|}{|a^{2(m-1)}|}\frac{-b}{|a^2|a}p_\beta\right].
\end{aligned}
\tag{3.22}
$$

Now, we compare (3.22) with (2.11) to again find that the normalization is fixed by

$$\mathcal{N}=Z\,. \tag{3.23}$$

The boundary state also has an overlap with the state A considered previously, and defined by

$$|A_{\mu\nu\delta}\rangle = A_{\mu\nu\delta}\alpha^\mu_{-a}\tilde{\alpha}^\nu_{-b}\tilde{\alpha}^\delta_{-c}|0\rangle\,. \tag{3.24}$$

The overlap is then

$$
\begin{aligned}
\langle A_{\mu\nu\delta}|B\rangle &= \mathcal{N}\int d^2a d^2b\delta(|a^2|-|b^2|-1)\exp\left(-\frac{\alpha'}{2}x^\mu U_{\mu\nu}x^\nu\right)\times \\
&\quad\times\exp\left(-\sum_{n=1}^{\infty}\frac{\alpha'}{2}p_\mu M_{-n0}\frac{1}{n}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{n}}\right)_{\mu\nu}\bar{M}_{-n0}p_\nu\right)\times \\
&\quad\times A_{\mu\nu\delta}\sqrt{\frac{\alpha'}{2}}\left[\sum_n abM_{-n-a}\Lambda^{\mu\nu}(n)\bar{M}_{-n-b}\sum_m cM_{-m0}\Lambda^{\alpha\delta}(m)\bar{M}_{-m-c}p_\alpha+\right.
\end{aligned}
$$

$$+\sum_n acM_{-n-a}\Lambda^{\mu\delta}(n)\bar{M}_{-n-c}\sum_m bM_{-m0}\Lambda^{\alpha\nu}(m)\bar{M}_{-m-b}p_\alpha -$$

$$-\frac{\alpha'}{2}\sum_n aM_{-n-a}\Lambda^{\mu\alpha}(n)\bar{M}_{-n0}\sum_m bM_{-m0}\Lambda^{\beta\nu}(m)\bar{M}_{-m-b}\times$$

$$\left.\times\sum_l cM_{-l0}\Lambda^{\gamma\delta}(l)\bar{M}_{-l-c}p_\alpha p_\beta p_\gamma\right], \tag{3.25}$$

and we have introduced the notational simplification

$$\frac{1}{m}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{m}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{m}}\right)^{\mu\nu}=\Lambda^{\mu\nu}(m)\,. \tag{3.26}$$

We compare (2.13) with (3.25) in the same manner that we have for the tachyon and the massless states. To verify the consistency requires the observation that

$$\frac{1}{(k-1)!}\partial^k z^d(y)|_{y=0}=k\bar{M}_{-d-k}\,. \tag{3.27}$$

This shows that the result obtained from differentiating the Greens function, as deformed in the sigma model, gives the same result as the overlap of a closed string oscillator mode with the boundary state. Using this, direct comparison of (2.13) to (3.25) gives $\mathcal{N}=Z$ again. This shows that our boundary state $|B\rangle$ gives the correct overlap with any closed string state, either on shell or off shell, and has fixed the normalization to be equal to the partition function. The form of the matrices M is such that upon integration over $\mathrm{PSL}(2,R)$ the overlap between $|B\rangle$ and any state that is not level matched will vanish. This can be understood as the requirement that the vertex operators in the sigma model language be level matched.

4. Loop corrections to partition functions

In the previous sections we described a boundary state and verified that it correctly reproduces the overlap with single particle closed string states as calculated in a sigma model approach. Now we use the boundary state developed to calculate the open string one loop partition function. The method makes use of the duality between open string one loop and closed string tree level diagrams, so to obtain the one loop partition function we take the sum of all possible single closed string exchange weighted by the closed string propagator. This provides a specific realization of the proposal of [12] where the open string one loop amplitude has exactly the poles related to long distance propagation of massless modes.

We wish to calculate the open string one loop correction to the partition function. The disk level contribution was given in (2.6) and the one loop correction to this is given in the sigma model calculation by using the propagator on an annulus world-sheet [25]. The boundary state version of the calculation is

$$Z_{\mathrm{One\,loop}}=\int d^2ad^2b\delta(|a^2|-|b^2|-1)d^2a'd^2b'\delta(|a^{'2}|-|b^{'2}|-1)\times$$

$$\times\mathcal{N}^2\langle B_{a',b'}|\frac{1}{L_0+\tilde{L}_0-2}|B_{a,b}\rangle\,. \tag{4.1}$$

In the above, $|B_{a,b}\rangle$ is as given in (3.16). Note that this formulation will explicitly give factorization of the amplitude in the closed string channel. Using this, the calculation of $Z_{\text{One loop}}$ is straightforward. Using an integral representation for the propagator we find

$$Z_{\text{One loop}} = \int d^2a d^2b \delta(|a^2| - |b^2| - 1) d^2a' d^2b' \delta(|a'^2| - |b'^2| - 1) \int_0^\infty dt \times \\ \times \mathcal{N}^2 \langle B_{a',b'}| e^{-t(L_0+\tilde{L}_0-2)} |B_{a,b}\rangle \,. \tag{4.2}$$

We suppress the PSL$(2,R)$ integrals temporarily, and calculate the inner product

$$\begin{aligned}\langle B_{a',b'}| e^{-t(L_0+\tilde{L}_0-2)} |B_{a,b}\rangle = & \left\langle -i\frac{\alpha'}{2} U^{\mu\nu} x_\nu \right| \exp\left(\sum_{n=1,j,k=-\infty}^{\infty} \alpha_k^\mu \bar{M}^{(1)}_{-n-k} \Lambda_{\mu\nu}(n) M^{(1)}_{-nj} \tilde{\alpha}_j^\nu \right) \times \\ & \times \exp\left(-t \sum_{n\geq 1} \left(\alpha^\mu_{-n} \alpha_{n\mu} + \tilde{\alpha}^\mu_{-n} \tilde{\alpha}_{n\mu} \right) - \frac{t\alpha'}{4} p^\mu p_\mu \right) \times \\ & \times \exp\left(\sum_{n=1,j,k=-\infty}^{\infty} \alpha^\mu_{-k} M^{(2)}_{-n-k} \Lambda_{\mu\nu}(n) \bar{M}^{(2)}_{-nj} \tilde{\alpha}^\nu_{-j} \right) \times \\ & \times \left| -i\frac{\alpha'}{2} U^{\mu\nu} x_\nu \right\rangle .\end{aligned} \tag{4.3}$$

In this, we have denoted the dependence on a particular PSL$(2,R)$ transform by a superscript on the appropriate matrix, and abbreviated the gaussian term in x acting on the Fock space vacuum. The above expression can be simplified considerably by using the relation

$$e^A e^B = e^B \left(\prod_{n=1}^{\infty} e^{\frac{1}{n!}[A,\ldots,[A,B]]} \right) e^A \tag{4.4}$$

which holds when $[A,B]$, $[A,[A,B]]$ and all similarly nested commutators commute with each other and with B, but do not commute with A. This then gives

$$\begin{aligned}\langle B_{a',b'}| e^{-t(L_0+\tilde{L}_0-2)} |B_{a,b}\rangle = & \left\langle -i\frac{\alpha'}{2} U^{\mu\nu} x_\nu \right| \exp\left(\sum_{n=1,j,k=-\infty}^{\infty} \alpha_k^\mu \bar{M}^{(1)}_{-n-k} \Lambda_{\mu\nu}(n) M^{(1)}_{-nj} \tilde{\alpha}_j^\nu \right) \times \\ & \times \exp\left(-\frac{t\alpha'}{4} p^\mu p_\mu \right) \times \\ & \times \exp\left(\sum_{n=1,j,k=-\infty}^{\infty} \alpha^\mu_{-k} e^{-tk} M^{(2)}_{-n-k} \Lambda_{\mu\nu}(n) \bar{M}^{(2)}_{-nj} \tilde{\alpha}^\nu_{-j} e^{-tj} \right) \times \\ & \times \left| -i\frac{\alpha'}{2} U^{\mu\nu} x_\nu \right\rangle \\ = & \, e^{2t} \exp \sum_k \frac{1}{k} \delta^\nu_\mu \delta_{mn} \left(\left[M^{(2)}_{-m-l} l e^{-tl} \bar{M}^{(1)}_{-k-l} \Lambda^\mu_\alpha(k) \times \right.\right. \\ & \left.\left. \times M^{(1)}_{-k-r} r e^{-tr} \bar{M}^{(2)}_{-n-r} \Lambda^\alpha_\nu(n) \right]^k \right)^{\mu\nu}_{mn} F(x)\end{aligned} \tag{4.5}$$

with matrix multiplication implied within the sum for both the Lorentz and oscillator indices, and $F(x)$ a gaussian that depends on x, U, F and the transformation parameters. $F(x)$ is determined to be

$$
\begin{aligned}
&F(x) = \\
&= \exp p^\mu p^\nu \Bigg(-\frac{t\alpha'}{4} g_{\mu\nu} + \bar{M}^{(1)}_{-k0} \Lambda^\mu_\alpha(k) M^{(1)}_{-k-d} \times \\
&\qquad \times \frac{1}{1 - d e^{-td} \bar{M}^{(2)}_{-n-d} \Lambda^\alpha_\beta(n) M^{(2)}_{-n-l} l e^{-tl} \bar{M}^{(1)}_{-p-l} \Lambda^\beta_\gamma(p) M^{(1)}_{-p-q}} \times \\
&\qquad \times q e^{-tq} \bar{M}^{(2)}_{-r-q} \Lambda^\gamma_\nu(q) M^{(2)}_{-q0} + \bar{M}^{(1)}_{-k0} \Lambda^\mu_\alpha(k) M^{(1)}_{-k-d} \times \\
&\qquad \times \frac{1}{1 - d e^{-td} \bar{M}^{(2)}_{-n-d} \Lambda^\alpha_\beta(n) M^{(2)}_{-n-l} l e^{-tl} \bar{M}^{(1)}_{-p-l} \Lambda^\beta_\nu(p) M^{(1)}_{-p-0}} + \\
&\qquad + \frac{1}{1 - \bar{M}^{(2)}_{-n0} \Lambda^\mu_\beta(n) M^{(2)}_{-n-l} l e^{-tl} \bar{M}^{(1)}_{-p-l} \Lambda^\beta_\gamma(p) M^{(1)}_{-p-q}} q e^{-tq} \bar{M}^{(2)}_{-r-q} \Lambda^\gamma_\nu(q) M^{(2)}_{-q0} \Bigg),
\end{aligned} \tag{4.6}
$$

with $p^\mu = -\frac{i\alpha'}{2} U^\mu_\nu x^\nu$ as in (3.4). It has the effect of localizing the interaction between two d-branes. The final expression for $Z_{\text{One loop}}$ is obtained by integrating (4.5) over the elements of the conformal transformations noted above. It is interesting to compare (4.5) with the calculation of the annulus amplitude from [12]. This analysis captures a different aspect of the closed string dynamics because rather than explicitly exhibiting the closed string poles for each mass level, this gives the sum over all closed string channels.

It is instructive to examine the form of (4.5). First, note that ignoring the x dependence we express

$$
\begin{aligned}
Z_{\text{One loop}} = &\int d^2 a d^2 b \delta(|a^2| - |b^2| - 1) d^2 a' d^2 b' \delta(|a'^2| - |b'^2| - 1) \times \\
&\times \mathcal{N}^2 e^{2t} \frac{1}{\det\left(1 - b e^{-tb} \bar{M}^{(2)}_{-n-b} \Lambda^\alpha_\beta(n) M^{(2)}_{-n-a} a e^{-ta} \bar{M}^{(1)}_{-p-a} \Lambda^\beta_\gamma(p) M^{(1)}_{-p-q}\right)}.
\end{aligned} \tag{4.7}
$$

In the cases of $U \to 0$ and $U \to \infty$ the matrices $\bar{M}\Lambda M$ revert to a particularly simple form. We have

$$
\begin{aligned}
\sum_{n\geq 1} \bar{M}_{-n-b} \Lambda^{\mu\nu}(n) M_{-n-a}\Big|_{(U\to 0)} &= \sum_{n\geq 1} \bar{M}_{-n-b} \frac{1}{n} \left(\frac{g - 2\pi\alpha' F}{g + 2\pi\alpha' F}\right)^{\mu\nu} M_{-n-a} \\
&= \left(\frac{g - 2\pi\alpha' F}{g + 2\pi\alpha' F}\right)^{\mu\nu} \frac{1}{b} \delta_{ab}
\end{aligned} \tag{4.8}
$$

and also

$$
\sum_{n\geq 1} \bar{M}_{-n-b} \Lambda^{\mu\nu}(n) M_{-n-a}\Big|_{(U\to\infty)} = -g^{\mu\nu} \frac{1}{b} \delta_{ab} \tag{4.9}
$$

which follow from the definition of M. We can see that in the case of $U = 0$ that the boundary state for a background gauge field localized on some extended object is recovered, and in the case $U = \infty$ the boundary state for Dirichlet boundary conditions is recovered, because the terms (4.8) and (4.9) appear in the exponentials as the coefficients of $-\alpha^{\mu}_{-a}\tilde{\alpha}^{\nu}_{-b}$ [21, 33]. In both the cases the integral over the volume of PSL$(2,R)$ becomes a trivial prefactor, as expected.

Evaluation of (4.7) in the limit $U \to 0$ gives

$$Z_{\text{One loop}}(U=0) = \prod_{r=1}^{\infty}\left(1-e^{-2t}\right)^{-D}\prod_{r=1}^{\infty}\det\left(g - \frac{1+e^{-2t}}{1-e^{-2t}}4\pi\alpha' F + 4\pi^2\alpha'^2F^2\right)^{-1} \tag{4.10}$$

exactly the result obtained in [25].

It is also possible to say something about the more general case. The sum $M^{(2)}_{-n-a}a \times e^{-ta}\bar{M}^{(1)}_{-q-a}$ can be taken over a and we find the result is a transformation in SL(2,C), of which PSL$(2,R)$ is a subgroup. Explicitly

$$\sum_{a\geq 1} M^2_{-n-a}ae^{-ta}\bar{M}^{(1)}_{-q-a} = \oint \frac{dz}{2\pi i}\frac{1}{z^n}\frac{\left[z(\bar{a}_1e^{-t/2}a_2 - b_1e^{t/2}\bar{b}_2) + (\bar{a}_1e^{-t/2}b_2 - b_1e^{t/2}\bar{a}_2)\right]^{q-1}}{\left[z(a_1e^{t/2}\bar{b}_2 - \bar{b}_1e^{-t/2}a_2) + (a_1e^{t/2}\bar{a}_2 - \bar{b}_1e^{-t/2}b_2)\right]^{q+1}}. \tag{4.11}$$

The fact that the integral over conformal factors becomes an integral over a transformation in a larger group is appropriate. The integral over t stands in the place of the integral over the Teichmuller parameter of the annulus, and the other degrees of freedom correspond to reparametrizations of the two ends of the annulus.

It is also interesting to examine how these expressions for $Z_{\text{One loop}}$ vary with U around the two fixed points of renormalization group flow, $U=0$ and $U=\infty$. From the tree level analysis [2] we know that the coupling U has an unstable fixed point at 0 and a stable fixed point at ∞ corresponding to Neumann and Dirichlet boundary conditions respectively. The action for the boundary fields is given in general by [1]

$$S = \left(\beta_i\frac{\partial}{\partial\lambda^i}+1\right)Z\,, \tag{4.12}$$

where β_i is the beta function for the coupling λ^i (in this case a coupling of interest is $\lambda = U$), and using this observation we can show that the one loop contribution to the partition function does not change the stability properties obtained by simply considering the tree level result.

We ignore the terms which go as $\sqrt{1/U}$ in the partition function because they produce a divergence associated with the volume of the unstable brane, and consider $Z_{\text{One loop}}$ near $U=0$ is

$$Z_{\text{One loop}} = Z_{\text{One loop}}(U=0) + \text{Tr}\left(U\frac{\partial}{\partial U}Z_{\text{One loop}}(U=0)\right) + \cdots\,. \tag{4.13}$$

The linear term in U is given by

$$\text{Tr}\left(U\frac{\partial}{\partial U}Z_{\text{One loop}}(U=0)\right) = \int d^2ad^2b\delta(|a^2|-|b^2|-1)\mathcal{V}_{\text{PSL}(2,R)}\mathcal{N}^2e^{2t}\times$$
$$\times\frac{1}{\det\left(1-e^{-2tb}\left(\frac{g-2\pi\alpha'F}{g+2\pi\alpha'F}\right)^2\right)}\times$$

$$
\begin{aligned}
&\times \operatorname{Tr}\left(\frac{(g-2\pi\alpha' F)/(g+2\pi\alpha' F)}{(g+2\pi\alpha' F)^2-(g-2\pi\alpha' F)^2}U\right)\times\\
&\times\frac{-4be^{-2b}}{1-e^{-2b}}\bar{M}_{-n-b}\frac{1}{n^2}M_{-n-b}.\\
=&\int d^2a d^2b\,\delta(|a^2|-|b^2|-1)\mathcal{V}_{\mathrm{PSL}(2,R)}\mathcal{N}^2e^{2t}\times\\
&\times\frac{1}{\det\left(1-e^{-2tb}\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)^2\right)}\times\\
&\times \operatorname{Tr}\left(\frac{(g-2\pi\alpha' F)/(g+2\pi\alpha' F)}{(g+2\pi\alpha' F)^2-(g-2\pi\alpha' F)^2}U\right)\sum_{n,b\geq 1}\frac{-4be^{-2b}}{1-e^{-2b}}\times\\
&\times\sum_{q=0}^{\min(n-1,b-1)}\frac{1}{(nb)^2}\left(\frac{b+n-q-1!}{q!n-q-1!b-q-1!}\right)^2\times\\
&\times\left(\frac{|b^2|}{|a^2|}\right)^{b+n-2q-2}\frac{1}{|a^2|^2}\,. \qquad (4.14)
\end{aligned}
$$

The critical point is not the form of the expression, but that the sign of (4.14) matches that obtained by varying the oscillator portion of (2.5) indicating the same instability.

A similar calculation can be done around the d-brane ($U\to\infty$) and we find that

$$
\begin{aligned}
\operatorname{Tr}\left(\frac{1}{U}\frac{\partial}{\partial(1/U)}Z_{\text{One loop}}\right)_{\frac{1}{U}=0} &= \int d^2ad^2b\,\delta(|a^2|-|b^2|-1)\mathcal{V}_{\mathrm{PSL}(2,R)}\frac{\mathcal{N}^2e^{2t}}{\det\left(1-e^{-2tb}\right)}\times\\
&\times 4\operatorname{Tr}\left(\frac{1}{U}\right)\sum_{n,b\geq 1}\frac{be^{-2b}}{1-e^{-2b}}\bar{M}_{-n-b}M_{-n-b}\,. \qquad (4.15)
\end{aligned}
$$

The expression $\sum_n \bar{M}_{-n-a}M_{-n-b}$ has zero entries on diagonal ($a=b$), so this variation vanishes about the condensate. This comparison between (4.14) and (4.15) shows that the case of Neumann boundary conditions, (corresponding to $U\to 0$) is unstable with respect to variations of the tachyon condensate since the linear variation does not vanish, but that Dirichlet boundary conditions, obtained as $U\to\infty$ are stable, as would be expected on general grounds [34].

5. Conclusions

In this note we have presented a generalization of the boundary state formalism that allows us to calculate the overlap of the boundary state with arbitrary on and off shell closed string states. We have shown that it exactly reproduces the calculations that would be done in a sigma model for an appropriate vertex operator, verifying the conjecture of [24], and this state is an off-shell interpolation between Neumann and Dirichlet boundary conditions for the open string. This generalization gives a prescription for the calculation of the open string one loop partition function for a given set of boundary interactions, and this prescription reproduces the expected modular transformation structure and explicitly factorizes in the closed string channel.

JHEP11(2003)021

Acknowledgments

The authors are grateful to Emil Akhmedov and Taejin Lee for many helpful conversations on this work. This work supported in part by the Canadian Natural Science and Engineering Research Council.

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